

Mathematical Analysis I

by

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Errata

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This list of errata records substantive changes made since the version of May 20, 2004.

Version of March 10, 2009

(1) Chapter 2, §§1–4, Axiom III of the real numbers, change

$$(\forall x, y, z \in E^1) \quad (x + y) + z = x + (y + x) \text{ and } (xy)z = x(yz).$$

to

$$(\forall x, y, z \in E^1) \quad (x + y) + z = x + (y + z) \text{ and } (xy)z = x(yz).$$

(2) Chapter 2, §§5–6, rewrite Problem 8 to be

8. For $n > 0$ define

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Verify *Pascal's law*,

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}.$$

Then prove by induction on n that

(i) $(\forall k \mid 0 \leq k \leq n) \binom{n}{k} \in N$; and

(ii) for any field elements a and b ,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}, \quad n \in N \text{ (the binomial theorem).}$$

(3) Chapter 2, §13, last displayed formula of the proof of Theorem 1, change

$$q_n = \sup A_n \leq a.$$

to

$$(\forall n > n_0) \quad q_n = \sup A_n \leq a.$$

(4) Chapter 3, §9, Example (d), change

with each map $f: A \rightarrow W$ treated as a single “vector” in W

to

with each map $f: A \rightarrow V$ treated as a single “vector” in W .

(5) Chapter 3, §13, Problem 9, change

In case (i), show that globes of radius ≤ 1 are the same under ρ and ρ' . In case (ii), prove that any $G_p(\epsilon)$ in (S, ρ) is also a globe in (S, ρ') of radius

$$\frac{\epsilon}{1 + \epsilon},$$

and any globe of radius < 1 in (S, ρ') is also a globe in (S, ρ) . (Find the converse formula as well!)

to

In case (i), show that globes $G_p(\epsilon)$ of radius $\epsilon \leq 1$ are the same under ρ and ρ' . In case (ii), prove that any $G_p(\epsilon)$ in (S, ρ) is also a globe $G_p(\epsilon')$ in (S, ρ') of radius

$$\epsilon' = \frac{\epsilon}{1 + \epsilon},$$

and any globe of radius $\epsilon' < 1$ in (S, ρ') is also a globe in (S, ρ) . (Find the converse formula for ϵ as well!)

(6) Chapter 3, §11, hint to Problem 10, change

$$\rho'_{13} \leq \rho'_{12} + \rho'_{23} \quad \text{and} \quad \rho''_{13} \leq \rho''_{22} + \rho''_{23}.$$

to

$$\rho'_{13} \leq \rho'_{12} + \rho'_{23} \quad \text{and} \quad \rho''_{13} \leq \rho''_{12} + \rho''_{23}.$$

(7) Chapter 3, §12, Problem 3, change

Prove that if $\bar{p} \in G_{\bar{q}}(r)$ in E^n , then $G_{\bar{q}}(r)$ contains a cube $[\bar{c}, \bar{d}]$ with center \bar{p}

to

Prove that if $\bar{p} \in G_{\bar{q}}(r)$ in E^n , then $G_{\bar{q}}(r)$ contains a cube $[\bar{c}, \bar{d}]$ with $\bar{c} \neq \bar{d}$ and with center \bar{p} .

(8) Chapter 3, §14, Problem 10, change

$$\left\{ 1, 1, \frac{1}{2}, 1\frac{1}{2}, \dots, \frac{1}{n}, 1 + \frac{1}{n}, \dots \right\}$$

to

$$\left\{ 1, 2, \frac{1}{2}, 1\frac{1}{2}, \dots, \frac{1}{n}, 1 + \frac{1}{n}, \dots \right\}$$

(9) Chapter 3, §15, hint to Problem 38, change

[Hint: Take limits of both sides in $a_{n+1} = \frac{1}{2}(a_n + b_n)$ to get $p = \frac{1}{2}(p + q)$.]

to

[Hint: Take limits of both sides in $b_{n+1} = \frac{1}{2}(a_n + b_n)$ to get $q = \frac{1}{2}(p + q)$.]

(10) Chapter 3, §16, Problem 15, change “Find such *quantifier formulas* for $p \in A$ ” to “Find such *quantifier formulas* for $p \in \bar{A}$ ”.

Thanks to the students of Math 504, Spring 2009, at Purdue for their comments.

Version of November 19, 2007

(1) Chapter 3, §15, Problem 25. In part (e), change

$$x_n = \sqrt{\sum_{k=1}^m a_k^n}, \text{ with } a_k > 0$$

to

$$x_n = \sqrt[n]{\sum_{k=1}^m a_k^n}, \text{ with } a_k > 0.$$

In the hint to part (a), change

$$0 < x_n < n^q \left[\left(1 + \frac{1}{n} \right)^q - 1 \right] < n^q$$

to

$$0 < x_n = n^q \left[\left(1 + \frac{1}{n} \right)^q - 1 \right] < n^q.$$

Thanks to Jonathan Ferron for his comments.

Version of May 14, 2007.

- (1) Chapter 4, §5, statement of Theorem 1. Change

$$f(p^+) = \inf_{a < x < b} f(x) \text{ for } p \in [a, b)$$

to

$$f(p^+) = \inf_{p < x < b} f(x) \text{ for } p \in [a, b).$$

- (2) Chapter 4, §13, before Theorem 1. Change “We now consider absolute convergence for *complete* range spaces” to “For the rest of this section we consider only *complete* range spaces.”
- (3) Chapter 4, §13, proof of Theorem 5. Change

we similarly obtain $|a_{n+1}| < |a_n|r$; hence by induction,

$$(\forall n \in N) \quad |a_n| \leq |a_1|r^{n-1}.$$

to

we similarly obtain $(\exists m) (\forall n \geq m) |a_{n+1}| < |a_n|r$; hence by induction,

$$(\forall n \geq m) \quad |a_n| \leq |a_m|r^{n-m}.$$

- (4) Chapter 4, §13, Example (C). Change

$$\overline{\lim} \sqrt[n]{a_n} = \lim \sqrt[2^n]{2^{-n}} = \frac{1}{\sqrt{2}} < 1$$

to

$$\overline{\lim} \sqrt[n]{a_n} = \lim \sqrt[2^{n-1}]{2^{-n}} = \frac{1}{\sqrt{2}} < 1.$$

- (5) Chapter 5, §1, Problem 7(i). Add the hint “For the converse, start with Problem 14(iii) of Chapter 4, §2.”
- (6) Chapter 5, §2, proof of Theorem 1. Change the second paragraph and the beginning of the third paragraph to

Thus $M = f(a)$ or $M = f(b)$; for the moment we assume $M = f(b)$ and $m = f(a)$. We must have $m < M$, for $m = M$ would make f *constant* on $[a, b]$, implying $f' = 0$. Thus $m = f(a) < f(b) = M$.

Now let $a \leq x < y \leq b$. Applying the previous argument to each of the intervals $[a, x]$, $[a, y]$, $[x, y]$, and $[x, b]$ (now using that $m = f(a) < f(b) = M$), we find that

$$f(a) \leq f(x) < f(y) \leq f(b). \quad (\text{Why?})$$

Thanks to the students of the Spring 2007 section of MA 504 for their comments.

Version of April 22, 2007

- (1) Chapter 3, §14, displayed equation just after Corollary 1. Change

$$G_p(\epsilon) \cap G_p(\epsilon)$$

to

$$G_p(\epsilon) \cap G_q(\epsilon).$$

- (2) Chapter 4, §1, statement of Corollary 1. Change “ $\lim_{x \rightarrow \rho} f(x)$ ” to “ $\lim_{x \rightarrow p} f(x)$ ”.
- (3) Chapter 4, §2, statement of Theorem 1. Change

$$f: A \rightarrow (T, \rho), \text{ with } A \subseteq (S, \rho)$$

to

$$f: A \rightarrow (T, \rho'), \text{ with } A \subseteq (S, \rho).$$

- (4) Chapter 4, §4, Note 1. Change

(i) $\frac{(\pm\infty)}{0^+} = \pm\infty$, $\frac{(\pm\infty)}{0^-} = \mp\infty$. If $q > 0$, then $\frac{q}{0^+} = +\infty$ and $\frac{q}{0^-} = -\infty$.

(ii) $\frac{\infty}{0} = \infty$.

(iv) $\frac{q}{\infty} = 0$.

(v) If $0 < |q| < 1$, then $q^{-\infty} = \infty$ and $q^{+\infty} = 0$.

(vi) If $|q| > 1$, then $q^{+\infty} = \infty$ and $q^{-\infty} = 0$.

to

(i) $\frac{(\pm\infty)}{0^+} = \pm\infty$, $\frac{(\pm\infty)}{0^-} = \mp\infty$.

(ii) If $q > 0$, then $\frac{q}{0^+} = +\infty$ and $\frac{q}{0^-} = -\infty$.

(iii) $\frac{\infty}{0} = \infty$.

(iv) $\frac{q}{\infty} = 0$.

- (5) Chapter 4, §5, statement of Theorem 3. Change “If $f: A \rightarrow E^*$ is monotone on a finite or infinite interval (a, b) containing $A \dots$ ” to “If $f: A \rightarrow E^*$ is monotone on a finite or infinite interval (a, b) contained in $A \dots$ ”
- (6) Chapter 4, §6, proof of Theorem 5. Change “Therefore, by the definition of completeness (Chapter 3, §17), $\{x_m\}$ has a limit $p \in E_1$ ” to “Therefore, by the definition of completeness (Chapter 3, §17), $\{x_m\}$ has a limit $p \in F_1$.”
- (7) Chapter 4, §8, statement of Theorem 1. Change “a function $f: A \rightarrow (T, \rho)$ ” to “a function $f: A \rightarrow (T, \rho')$ ” and “Briefly, any continuous image $f[B]$ of a compact

set B is compact itself” to “Briefly, the continuous image of a compact set is compact.”

- (8) Chapter 4, §8, Problem 12. Change

$$(f \vee g)(x) = \max((f(x), f(g)))$$

to

$$(f \vee g)(x) = \max((f(x), g(x))).$$

- (9) Chapter 4, §9, proof of Theorem 1. Add a paragraph break before “We denote this particular closed segment by $L[\bar{p}_1, \bar{q}_1], \dots$ ”
- (10) Chapter 4, §11, Definition 2, change “A function $f: (X \times Y) \rightarrow (T, \rho) \dots$ ” to “A function $f: (X \times Y) \rightarrow (T, \rho') \dots$ ”
- (11) Chapter 4, §11, statement of Theorem 2. Change “Let (T, ρ) be complete” to “Let (T, ρ') be complete.”
- (12) Chapter 4, §12, Example (a), footnote 2. Keep footnote on page with footnote mark.
- (13) Chapter 4, §12, Problem 3(iv). Change “ $f_n(x) = \frac{nx^2}{1+nx}$ ” to “ $f_n(x) = \frac{x}{1+nx}$.”
- (14) Chapter 4, §13, statement of Theorem 1, footnote 4. Keep footnote on page with footnote mark.

Thanks to the students of the Spring 2007 section of MA 504 for their comments.

Version of February 19, 2007

- (1) Chapter 2, §§8–9, end of proof of Theorem 1. Change

$$p = \text{glb } A = \inf B. \quad \square$$

to

$$p = \text{glb } A = \inf A. \quad \square$$

- (2) Chapter 2, §13, after Corollary 1. Change

$$\begin{aligned} \bar{L} &\geq p_n = \inf A_n \geq \inf A_1 = \inf_n x_n \text{ and} \\ \bar{L} &\leq q_n = \sup A_n \leq \sup A_1 = \sup_n x_n, \end{aligned}$$

to

$$\begin{aligned} \underline{L} &\geq p_n = \inf A_n \geq \inf A_1 = \inf_n x_n \text{ and} \\ \bar{L} &\leq q_n = \sup A_n \leq \sup A_1 = \sup_n x_n, \end{aligned}$$

- (3) Chapter 2, §13, proof of Theorem 2. Change “Now, exactly as in the proof of Theorem 2, one *excludes*

$$q \neq \underline{\lim} x_n \text{ and } q \neq \underline{\lim} x_n.”$$

to “Now, exactly as in the proof of Theorem 2, one *excludes*

$$q \neq \underline{\lim} x_n \text{ and } q \neq \overline{\lim} x_n.”$$

- (4) Chapter 3, §8, just before Definition 1. Change “In particular, if $z = (x, y)$ and $z' = (x', y')$, we have ...” to “In particular, if $z = (x, y)$ and $z' = (x', y')$, we have ...”.
- (5) Chapter 3, §8, Problem 7. Change “Prove that if $z = z'z''$, then ...” to “Prove that $z = z'z''$ if ...”.
- (6) Chapter 3, §12, hint to Problem 11. Change

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n} \right] = (0, 1).$$

to

$$\bigcup_{n=2}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n} \right] = (0, 1).$$

- (7) Chapter 3, §13, Problem 9(d). Change

$$\frac{n(n-1)}{(n-2)^2}$$

to

$$\frac{n(n-1)}{(n+2)^2}.$$

- (8) Chapter 3, §15, Theorem 1, statement of part (iii). Change

$$\frac{x_m}{a_m} \rightarrow \frac{q}{a} \text{ if } a \neq 0.$$

to

$$\frac{x_m}{a_m} \rightarrow \frac{q}{a} \text{ if } a \neq 0 \text{ and for all } m \geq 1, a_m \neq 0.$$

- (9) Chapter 3, §15, Example. Change

$$\begin{aligned} 2 < x_n &< 2 + \frac{1}{2!} + \cdots + \frac{1}{n!} \\ &\leq 2 + \frac{1}{2} + \cdots + \frac{1}{2^n} = 2 + \frac{1}{2} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} < 2 + 1 = 3. \end{aligned}$$

to

$$\begin{aligned} 2 < x_n < 2 + \frac{1}{2!} + \cdots + \frac{1}{n!} &\leq 2 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \\ &= 2 + \frac{1}{2} \left(1 + \cdots + \frac{1}{2^{n-2}} \right) = 2 + \frac{1}{2} \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{\frac{1}{2}} < 2 + 1 = 3. \end{aligned}$$

- (10) Chapter 3, §15, Corollary 1. Change “Suppose $\lim s_m = p$ ” to “Suppose $\lim x_m = p$ ”.

Thanks to the students of the Spring 2007 section of MA 504 for their comments.

Version of January 5, 2007

- (1) Chapter 5, §6, Proof of Theorem 1'. Change references to Cauchy's law of the mean to “Theorem 2 of §2” from “Theorem 3 of §2”.
- (2) Chapter 5, §6, Example (a). Change

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta_n x}}{(n+1)!}, \quad 0 < \theta_n < 1. \quad (8)$$

to

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta_n x}}{(n+1)!} x^{n+1}, \quad 0 < \theta_n < 1. \quad (8)$$

Indicate that the remainder R_n is positive when $x > 0$.

- (3) Chapter 5, §7, end of page 300. Change

$$|q_i - q| + |q - q_{i-1}| \geq |q_i - q_{i-1}| \quad (\text{triangle law})$$

to

$$|q_i - q| + |q - q_{i-1}| \geq |q_i - q_{i-1}| \quad (\text{triangle law}).$$

Thanks to Daniele Pala for the changes.

Version of July 26, 2006

- (1) Chapter 4, §12, Problem 14. Part (iii), change “ $\lim_{x \rightarrow -\infty} \frac{a^x}{x^q} = 0$ ” to “ $\lim_{x \rightarrow \infty} \frac{a^{-x}}{x^q} = 0$ ”.
- Part (iv), change “ $\lim_{x \rightarrow -\infty} \frac{a^x}{x^q} = +\infty$ ” to “ $\lim_{x \rightarrow \infty} \frac{a^{-x}}{x^q} = +\infty$ ”.
- (2) Chapter 4, §13, proof of Theorem 5. Refer to Problem 17, not Problem 18, in Chapter 4, §12.

- (3) Chapter 5, §1, Example (b). Change “ $f'(x) = \sin x - i \cdot \cos x$ ” to “ $f'(x) = -\sin x + i \cdot \cos x$ ”.
- (4) Chapter 5, §4, proof of Lemma 1. Change “ $|f(x) - f(x)| \leq \dots$ ” to “ $|f(x) - f(r)| \leq \dots$ ”.
- (5) Chapter 5, §5, Note 2. Change “ $G(a) = F(a) - f(a)$ ” to “ $G(a) = F(a) - F(a)$ ”.
- (6) Chapter 5, §5, proof of Corollary 1. Change “as are f and g ” to “as are F and G .”

Thanks to Daniele Pala for the changes.

Version of July 25, 2006:

- (1) Chapter 3, §12, Problem 11, in the hint, change

$$\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

to

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$